Please check the examination details below before entering your candidate information

Candidate surname	Other names

Pearson Edexcel Level 3 GCE

Centre Number

Candidate Number

Tuesday 23 June 2020

Afternoon (Time: 1 hour 30 minutes) | Paper Reference **9FM0/4A**

Further Mathematics
Advanced
Paper 4A: Further Pure Mathematics 2

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B)
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 8 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over ▶

1/1/1/1/1/

Pearson

1. A small sports club has 12 adult members and 14 junior members.

The club needs to enter a team of 8 players for a particular competition.

Determine the number of ways in which the team can be selected if

(i) there are no restrictions on the team,

(1)

(ii) the team must contain 4 adults and 4 juniors,

(2)

(iii) more than half the team must be adults.

(3)

i. There are 26 members in total

We are selecting 8 members from 26 so $\binom{26}{8}$

$\binom{26}{8} = 1562275$

1562275

ii. There are 12 adults - we need to choose 4 from them : $\binom{12}{4}$

There are 14 juniors - we need to choose 4 from them : $\binom{14}{4}$

$\binom{12}{4} \times \binom{14}{4}$

$= 495 \times 1001$

$= 495495$

495495

iii. needs at least 5 adults

5 adults + 3 juniors $\binom{12}{5} \times \binom{14}{3} = 288288$

6 adults + 2 juniors $\binom{12}{6} \times \binom{14}{2} = 84084$

7 adults + 1 juniors $\binom{12}{7} \times \binom{14}{1} = 11088$

8 adults + 0 juniors $\binom{12}{8} \times \binom{14}{0} = 495$

all the possibilities

$288288 + 84084 + 11088 + 495$

$= 383955$

4.3: Solving 2nd Order Recurrence Relations

2. Solve the recurrence system

$$u_1 = 1 \qquad u_2 = 4$$
$$9u_{n+2} - 12u_{n+1} + 4u_n = 3n$$

(9)

Associated homogenous recurrence relation:

$9u_{n+2} - 12u_{n+1} + 4u_n = 0$

$9r^2 - 12r + 4 = 0$

$(3r-2)^2 = 0$

$\Rightarrow r = 2/3$ (repeated root)

So the complementary function is:

$u_n = (A + Bn)\left(\frac{2}{3}\right)^n$

Try the particular integral (P.I.): $u_n = \lambda n + \mu$

$u_{n+1} = \lambda(n+1) + \mu$

$u_{n+2} = \lambda(n+2) + \mu$

$9\{\lambda(n+2) + \mu\} - 12\{\lambda(n+1) + \mu\} + 4\{\lambda n + \mu\} = 3n$

$9\lambda n + 18\lambda + 9\mu - 12\lambda n - 12\lambda - 12\mu + 4\lambda n + 4\mu = 3n$

$\lambda n + 6\lambda + \mu = 3n$

$\lambda n + 6\lambda + \mu = 3n + 0$

$\lambda = 3$

$6\lambda + \mu = 0$

equate coefficients

sub $\lambda = 3$ into $6\lambda + \mu = 0$ to find μ

$6(3) + \mu = 0$

$18 + \mu = 0$

$\mu = -18$

P.I: $3n - 18$

gen sol^n: C.F. + P.I

$u_n = (A + Bn)\left(\frac{2}{3}\right)^n + 3n - 18$

Question 2 continued

Given boundary conditions: $u_1 = 1$

$u_2 = 4$

Sub into gen soln eqn and form 2 eqns containing A and B - solve simultaneously.

$u_1 = 1$

$1 = (A+B)\left(\frac{2}{3}\right)^1 + 3(1) - 18$

$1 = \frac{2}{3}A + \frac{2}{3}B - 15$

$\frac{2}{3}A + \frac{2}{3}B = 16$

$2A + 2B = 48$

$A + B = 24$ ①

$u_2 = 4$

$4 = (A+2B)\left(\frac{2}{3}\right)^2 + 3(2) - 18$

$4 = (A+2B)\left(\frac{4}{9}\right) + 6 - 18$

$4 = \frac{4}{9}A + \frac{8}{9}B - 12$

$\frac{4}{9}A + \frac{8}{9}B = 16$

$4A + 8B = 144$

$A + 2B = 36$ ②

① − ② :

$A + B = 24$

$A + 2B = 36$ ⊖

$-B = -12$

$B = 12$

sub in b value into ① to find a

$A + (12) = 24$

$A = 24 - 12 = 12$

$A = 12$

$A = 12$ & $B = 12$

Particular soln: $u_n = (12 + 12n)\left(\frac{2}{3}\right)^n + 3n - 18$ //

5.1: Eigenvalues & Eigenvectors
5.3: The Cayley-Hamilton Theorem

3.

$$\mathbf{M} = \begin{pmatrix} 1 & k & -2 \\ 2 & -4 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

where k is a constant.

(a) Show that, in terms of k, a characteristic equation for $\mathbf{M}$ is given by

$$\lambda^3 - (2k + 13)\lambda + 5(k + 6) = 0$$

(3)

Given that det $\mathbf{M} = 5$

(b) (i) find the value of k

(ii) use the Cayley-Hamilton theorem to find the inverse of $\mathbf{M}$.

(7)

a. characteristic eq^n: $\det(M - \lambda I) = 0$

$$M - \lambda I = \begin{pmatrix} 1 & k & -2 \\ 2 & -4 & 1 \\ 1 & 2 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M - \lambda I = \begin{pmatrix} 1 & k & -2 \\ 2 & -4 & 1 \\ 1 & 2 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$M - \lambda I = \begin{pmatrix} 1-\lambda & k & -2 \\ 2 & -4-\lambda & 1 \\ 1 & 2 & 3-\lambda \end{pmatrix}$$

$\det(M - \lambda I) = 0$

$$\det\left[\begin{pmatrix} 1-\lambda & k & -2 \\ 2 & -4-\lambda & 1 \\ 1 & 2 & 3-\lambda \end{pmatrix}\right] = 0$$

$$(1-\lambda)\begin{vmatrix} -4-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} - (k)\begin{vmatrix} 2 & 1 \\ 1 & 3-\lambda \end{vmatrix} + (-2)\begin{vmatrix} 2 & -4-\lambda \\ 1 & 2 \end{vmatrix} = 0$$

$$(1-\lambda)\left[(-4-\lambda)(3-\lambda) - (2)(1)\right] - (k)\left[(2)(3-\lambda) - (1)(1)\right] + (-2)\left[(2)(2) - (1)(-4-\lambda)\right] = 0$$

Question 3 continued

$(1-\lambda)[-12+\lambda+\lambda^2-2]-k[6-2\lambda-1]-2[4+4+\lambda]=0$

$(1-\lambda)(\lambda^2+\lambda-14)-k(-2\lambda+5)-2(\lambda+8)=0$

$\lambda^2+\lambda-14-\lambda^3-\lambda^2+14\lambda+2\lambda k-5k-2\lambda-16=0$

$-\lambda^3+(13+2k)\lambda-30-5k=0$

$-\lambda^3+(13+2k)\lambda-5(k+6)=0$ $\times -1$

$\lambda^3-(13+2k)\lambda+5(k+6)=0$

bi. $\det\begin{pmatrix}1 & k & -2\\ 2 & -4 & 1\\ 1 & 2 & 3\end{pmatrix}=5$

$(1)\begin{vmatrix}-4 & 1\\ 2 & 3\end{vmatrix}-(k)\begin{vmatrix}2 & 1\\ 1 & 3\end{vmatrix}+(-2)\begin{vmatrix}2 & -4\\ 1 & 2\end{vmatrix}=5$

$(1)\left[(-4)(3)-(2)(1)\right]-(k)\left[(2)(3)-(1)(1)\right]+(-2)\left[(2)(2)-(-4)(1)\right]=5$

$(1)(-14)-(k)(5)+(-2)(8)=5$

$-14-5k-16=5$

$-5k-30=5$

$-5k=35$

$k=-7$

$k=-7$

ii. using $k=-7$, can now change characteristic eq^n: $\lambda^3-(2k+13)\lambda+5(k+6)=0$

$\lambda^3-(2\{-7\}+13)\lambda+5(\{-7\}+6)=0$

$\lambda^3-(-1)\lambda+5(-1)=0$

$\lambda^3+\lambda-5=0$

Use Cayley-theorem:

Characteristic eq^n $\lambda^3+\lambda-5=0$ can be written as $M^3+M-5I=0$

Question 3 continued

$M^3 + M - 5I = 0$ $\div M$

$M^2 + I - 5M^{-1} = 0$

rearrange for M^{-1}

$5M^{-1} = M^2 + I$

$M^{-1} = \dfrac{M^2 + I}{5} = \frac{1}{5}(M^2 + I)$

find M^2: $\begin{pmatrix} 1 & -7 & -2 \\ 2 & -4 & 1 \\ 1 & 2 & 3 \end{pmatrix}^2 = \begin{pmatrix} -15 & 17 & -15 \\ -5 & 4 & -5 \\ 8 & -9 & 9 \end{pmatrix}$

use calculator to get directly

$M^{-1} = \frac{1}{5}\left[\begin{pmatrix} -15 & 17 & -15 \\ -5 & 4 & -5 \\ 8 & -9 & 9 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] = \frac{1}{5}\begin{pmatrix} -14 & 17 & -15 \\ -5 & 5 & -5 \\ 8 & -9 & 10 \end{pmatrix}$

$\boxed{M^{-1} = \frac{1}{5}\begin{pmatrix} -14 & 17 & -15 \\ -5 & 5 & -5 \\ 8 & -9 & 10 \end{pmatrix}}$ //

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4.

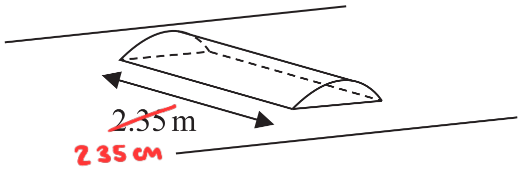


Figure 1

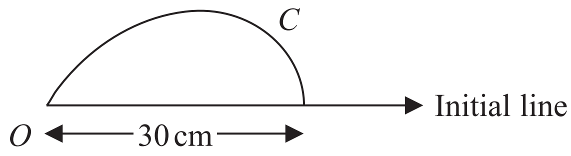


Figure 2

Figure 1 shows a sketch of a design for a road speed bump of width 2.35 metres. The speed bump has a uniform cross-section with vertical ends and its length is 30 cm. A side profile of the speed bump is shown in Figure 2.

The curve C shown in Figure 2 is modelled by the polar equation

$$r = 30(1 - \theta^2) \qquad 0 \leqslant \theta \leqslant 1$$

limits given - will use for integral

The units for r are centimetres and the initial line lies along the road surface, which is assumed to be horizontal.

Once the speed bump has been fixed to the road, the visible surfaces of the speed bump are to be painted.

Determine, in cm^2, the area that is to be painted, according to the model.

IS NOT asking for volume

(10)

Area of a sector:

$A = \frac{1}{2}\int r^2 \, d\theta$ (polar coordinates)

GIVEN IN FORMULAE BOOKLET

$r = 30(1 - \theta^2)$

$r^2 = [30(1-\theta^2)]^2$

$= 900(1-\theta^2)^2$

$= 900(\theta^4 - 2\theta^2 + 1)$

$A = \frac{1}{2}\int_0^1 900(\theta^4 - 2\theta^2 + 1) \, d\theta$

take 900 out of integral

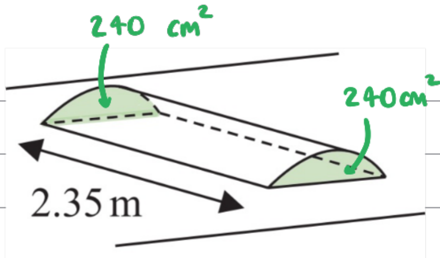
$A = 450\int_0^1 \theta^4 - 2\theta^2 + 1 \, d\theta$

$A = 450\left[\frac{\theta^5}{5} - \frac{2\theta^3}{3} + \theta\right]_0^1$

$A = 450\left[\left\{\frac{(1)^5}{5} - \frac{2(1)^3}{3} + (1)\right\} - \left\{\frac{(0)^5}{5} - \frac{2(0)^3}{3} + (0)\right\}\right]$

$A = 450\left(\frac{8}{15}\right) = 240$

Question 4 continued



For both sides, must do 2×240

$= 480 \text{ cm}^2$

Must now work out arc length of C:

$$S = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$ GIVEN IN FORMULAE BOOKLET

$r = 30(1-\theta^2)$

$r = 30 - 30\theta^2$

$\frac{dr}{d\theta} = -60\theta$

$\left(\frac{dr}{d\theta}\right)^2 = (-60\theta)^2 = 3600\theta^2$

$r^2 = 900(1-\theta^2)^2$

$$S = \int_0^1 \sqrt{900(1-\theta^2)^2 + 3600\theta^2}\, d\theta$$

↓ factor 900 out

$$S = \int_0^1 \sqrt{900\left[(1-\theta^2)^2 + 4\theta^2\right]}\, d\theta$$

$$S = \int_0^1 \sqrt{900\left[\theta^4 - 2\theta^2 + 1 + 4\theta^2\right]}\, d\theta$$

$$S = \int_0^1 \sqrt{900\left[\theta^4 + 2\theta^2 + 1\right]}\, d\theta$$

$$S = \int_0^1 \sqrt{900(\theta^2+1)^2}\, d\theta$$

↓ split surd into 2

$$S = \int_0^1 \sqrt{900}\,\cancel{\sqrt{\ }}(\theta^2+1)^{\cancel{2}}\, d\theta$$

$$S = \int_0^1 30(\theta^2+1)\, d\theta$$

↓ take 30 out integral

$$S = 30\int_0^1 \theta^2 + 1\, d\theta$$

$$S = 30\left[\frac{\theta^3}{3} + \theta\right]_0^1$$

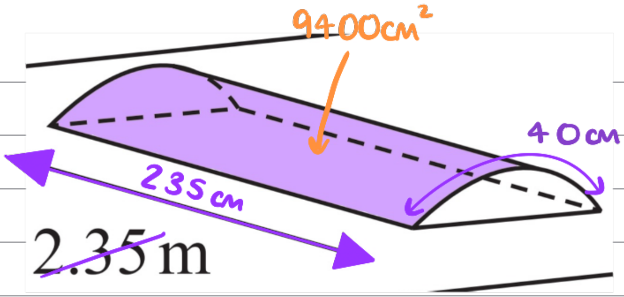
Turn over ▶

Question 4 continued

$S = 30\left[\left\{\frac{(1)^3}{3} + (1)\right\} - \left\{\frac{(0)^3}{3} + (0)\right\}\right]$

$S = 30\left(\frac{4}{3}\right)$

$S = 40\text{cm}$



To work out area of wide section: $40 \times 235 = 9400$

$9400 + 480$ – find total surface area of speed bump which can be painted

$= 9880\text{cm}^2$

9880cm^2

5. A transformation T from the z-plane to the w-plane is given by

$$w = \frac{1 - 3z}{z + 2\mathrm{i}} \qquad z \neq -2\mathrm{i}$$

The circle with equation $|z + \mathrm{i}| = 3$ is mapped by T onto the circle C.

(a) Show that the equation for C can be written as

$$3|w + 3| = |1 + (3 - w)\mathrm{i}|$$

(4)

(b) Hence find

(i) a Cartesian equation for C,

(ii) the centre and radius of C.

(6)

a. $w = \frac{1-3z}{z+2i}$

$w(z+2i) = 1-3z$

$wz + 2iw = 1-3z$

$wz + 3z = 1-2iw$

$z(w+3) = 1-2iw$

$z = \frac{1-2wi}{w+3}$

rearrange for z

Sub into $|z+i|=3$ eq^n

$\left|\frac{1-2wi}{w+3} + i\right| = 3$

$\left|\frac{(1-2wi)+i(w+3)}{w+3}\right| = 3$

$\left|\frac{1-2wi+wi+3i}{w+3}\right| = 3$

$\left|\frac{1-wi+3i}{w+3}\right| = 3$

Split using rule: $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

$\frac{|1-wi+3i|}{|w+3|} = 3$

multiply $|w+3|$ on both sides

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Question 5 continued

$|1-wi+3i| = 3|w+3|$

$|1+(3-w)i| = 3|w+3|$ //

bi. USE ALGEBRAIC APPROACH OF EVALUATING LOCI:

$3|w+3| = |1+(3-w)i|$

let $w = u+vi$

$3|u+vi+3| = |1+(3-\{u+vi\})i|$

$3|(u+3)+vi| = |1+(3-u-vi)i|$

$3|(u+3)+(v)i| = |1+3i-ui-vi^2|$ $\quad i^2=-1 \quad \therefore -vi^2 = (-v)(-1) = v$

$3|(u+3)+(v)i| = |(1+v)+(3-u)i|$

$3\sqrt{(u+3)^2+(v)^2} = \sqrt{(1+v)^2+(3-u)^2}$

$\left(3\sqrt{(u+3)^2+(v)^2}\right)^2 = \left(\sqrt{(1+v)^2+(3-u)^2}\right)^2$ square both sides to get rid of sqrt $\sqrt{\ }$

$(3)^2\left(\cancel{\sqrt{(u+3)^2+(v)^2}}\right)^{\cancel{2}} = \left(\cancel{\sqrt{(1+v)^2+(3-u)^2}}\right)^{\cancel{2}}$

$9\left[(u+3)^2+(v)^2\right] = (1+v)^2+(3-u)^2$

$9(u+3)^2+9v^2 = v^2+2v+1+u^2-6u+9$

$9(u^2+6u+9)+9v^2 = v^2+2v+1+u^2-6u+9$

$9u^2+54u+81+9v^2 = v^2+2v+1+u^2-6u+9$

$8u^2+60u+8v^2-2v+71=0$ $\quad \div 8$

$u^2+\frac{60}{8}u+v^2-\frac{2}{8}v+\frac{71}{8}=0$

$\left(u+\frac{15}{4}\right)^2-\frac{225}{16}+\left(v-\frac{1}{8}\right)^2-\frac{1}{64}+\frac{71}{8}=0$ complete the square for x & y

$\left(u+\frac{15}{4}\right)^2+\left(v-\frac{1}{8}\right)^2-\frac{333}{64}=0$

$\left(u+\frac{15}{4}\right)^2+\left(v-\frac{1}{8}\right)^2=\frac{333}{64}$

$\left(u+\frac{15}{4}\right)^2+\left(v-\frac{1}{8}\right)^2=\frac{333}{64}$ //

ii. centre: $\left(-\frac{15}{4}, \frac{1}{8}\right)$

radius: $\sqrt{\frac{333}{64}} = \frac{3\sqrt{37}}{8}$

2.2: Cayley Tables & Finite Groups
2.3: Order & Subgroups
2.4: Isomorphism

6.

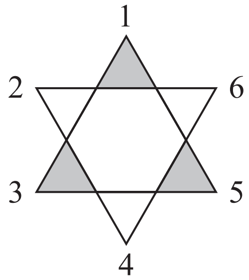


Figure 3

Figure 3 shows a plane shape made up of a regular hexagon with an equilateral triangle joined to each edge and with alternate equilateral triangles shaded.

The symmetries of this shape are the rotations and reflections of the plane that preserve the shape and its shading.

The symmetries of the shape can be represented by permutations of the six vertices labelled 1 to 6 in Figure 3. The set of these permutations with the operation of composition form a group, G.

(a) Describe geometrically the symmetry of the shape represented by the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 1 & 2 \end{pmatrix}$$

(2)

(b) Write down, in similar two-line notation, the remaining elements of the group G.

(4)

(c) Explain why each of the following statements is false, making your reasoning clear.

(i) G has a subgroup of order 4

(ii) G is cyclic.

(2)

Diagram 1, on page 23, shows an unshaded shape with the same outline as the shape in Figure 3.

(d) Shade the shape in Diagram 1 in such a way that the group of symmetries of the resulting shaded shape is isomorphic to the cyclic group of order 6

(2)

a. rotation about centre of shape, through an angle 120° anticlockwise

b. Group G consists of all symmetries of the shape - including rotations and reflections
- rotations: identity, 120° anticlockwise, 240° anticlockwise
- reflections: reflection through 1 & 4, reflection through 2 & 5, reflection through 3 & 6

Question 6 continued

Rotations:

identity - $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$

$120°$ anticlockwise - $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 1 & 2 \end{pmatrix}$ ← in part (a) so don't include

$240°$ anticlockwise - $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 1 & 2 & 3 & 4 \end{pmatrix}$

Reflections:

reflection through 1 & 4 - $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 5 & 4 & 3 & 2 \end{pmatrix}$

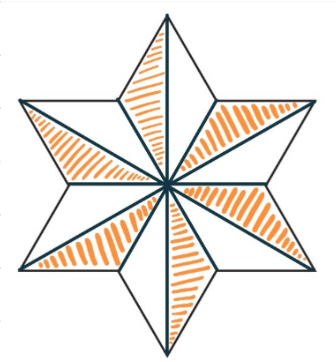
reflection through 2 & 5 - $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 1 & 6 & 5 & 4 \end{pmatrix}$

reflection through 3 & 6 - $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 3 & 2 & 1 & 6 \end{pmatrix}$

ci. G has order 6 so cannot have a subgroup of order 4 by Lagrange's theorem

ii. There is no element of order 6 that generates the group//

d. rotation must be preserved but reflection symmetry must be broken



7.

$$I_n = \int (4 - x^2)^{-n}\, dx \qquad n > 0$$

(a) Show that, for $n > 0$

$$I_{n+1} = \frac{x}{8n(4 - x^2)^n} + \frac{2n - 1}{8n} I_n$$

(5)

(b) Find I_2

(3)

a. $I_n = \int (4 - x^2)^{-n}\, dx$

$I_n = \int \underbrace{1}_{v'} \underbrace{(4 - x^2)^{-n}}_{u}\, dx$

Integration by parts:
$\int u \frac{dv}{dx}\, dx = uv - \int v \frac{du}{dx}\, dx$
Given in formulae booklet

$u = (4 - x^2)^{-n}$ $\qquad v' = 1$

$u' = -n(4 - x^2)^{-n-1}(-2x)$ $\qquad v = x$

$I_n = x(4 - x^2)^{-n} - \int -n(4 - x^2)^{-n-1}(-2x^2)\, dx$

since $(-n)(2) = -2n$ is a constant, take out of integral

$I_n = x(4 - x^2)^{-n} - (-2n)\int (4 - x^2)^{-n-1}(-x^2)\, dx$

$I_n = x(4 - x^2)^{-n} + 2n\int (4 - x^2)^{-n-1}(-x^2)\, dx$

rewrite $-x^2 = 4 - x^2 - 4$

$I_n = x(4 - x^2)^{-n} + 2n\int (4 - x^2)^{-n-1}(4 - x^2 - 4)\, dx$

split

$I_n = x(4 - x^2)^{-n} + 2n\int (4 - x^2)^{-n-1}(4 - x^2)^1 + (4 - x^2)^{-n-1}(-4)\, dx$

combine

$I_n = x(4 - x^2)^{-n} + 2n\int (4 - x^2)^{-n} - 4(4 - x^2)^{-(n+1)}\, dx$

split integral into 2

$I_n = x(4 - x^2)^{-n} + 2n\int (4 - x^2)^{-n}\, dx + 2n\int -4(4 - x^2)^{-(n+1)}\, dx$

take -4 out of integral

$I_n = x(4 - x^2)^{-n} + 2n\int (4 - x^2)^{-n}\, dx - 8n\int (4 - x^2)^{-(n+1)}\, dx$

$I_n = x(4 - x^2)^{-n} + 2nI_n - 8nI_{n+1}$

rewrite

$8nI_{n+1} = \dfrac{x}{(4 - x^2)^n} + 2nI_n - I_n$

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Question 7 continued

$8nI_{n+1} = \frac{x}{(4-x^2)^n} + 2nI_n - I_n$

group up I_n terms

$8nI_{n+1} = \frac{x}{(4-x^2)^n} + (2n-1)I_n$

$\times \frac{1}{8n}$ on both sides

$I_{n+1} = \frac{x}{8n(4-x^2)^n} + \frac{(2n-1)I_n}{8n}$ //

b. Use reduction formulae to find I_2:

$I_2 = \frac{x}{8(1)(4-x^2)^1} + \frac{2(1)-1}{8(1)} I_1$

Work out I_1 directly (Q states $n>0$)

$I_1 = \int (4-x^2)^{-1}\,dx = \int \frac{1}{4-x^2}\,dx$

Use formulae booklet:

$f(x)$	$\int f(x)\,dx$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert$
$a>0$	or $\frac{1}{a}\,\text{artanh}\left(\frac{x}{a}\right)$

$\int \frac{1}{4-x^2}\,dx = \frac{1}{2(2)}\ln\left\lvert\frac{2+x}{2-x}\right\rvert = \frac{1}{4}\ln\left\lvert\frac{2+x}{2-x}\right\rvert$

$a=2$

$I_2 = \frac{x}{8(4-x^2)} + \frac{1}{8}\left(\frac{1}{4}\ln\left\lvert\frac{2+x}{2-x}\right\rvert\right)$

$I_2 = \frac{x}{8(4-x^2)} + \frac{1}{32}\ln\left\lvert\frac{2+x}{2-x}\right\rvert + c$ //

Mark scheme also accepts $I_2 = \frac{x}{8(4-x^2)} + \frac{1}{16}\,\text{artanh}\left(\frac{x}{2}\right) + c$ if artanh version is used

8. The four digit number $n = abcd$ satisfies the following properties:

(1) $n \equiv 3 \pmod 7$

(2) n is divisible by 9

(3) the first two digits have the same sum as the last two digits

(4) the digit b is smaller than any other digit

(5) the digit c is even

(a) Use property (1) to explain why $6a + 2b + 3c + d \equiv 3 \pmod 7$ **(2)**

(b) Use properties (2), (3) and (4) to show that $a + b = 9$ **(4)**

(c) Deduce that $c \equiv 5(a - 1) \pmod 7$ **(2)**

(d) Hence determine the number n, verifying that it is unique. You must make your reasoning clear. **(4)**

a. can rewrite $n = abcd$ as $n = 1000a + 100b + 10c + d$

$n = 1000a + 100b + 10c + d$ take (mod 7) of everything

$n \equiv 6a + 2b + 3c + d \pmod 7$

since $n \equiv 3 \pmod 7$ ①

$\boxed{6a + 2b + 3c + d \equiv 3 \pmod 7}$ //

b. (2) if n is divisible by 9 - sum of digits must be divisible by 9

$a + b + c + d = 9, 18, 27, \underline{36}$

we can say 9 and not put 0 since otherwise code wouldn't exist

if $a,b,c,d = 9$ max value we can have is 36.

(3) $a + b = c + d$

(4) if $a + b =$ odd, $c + d \cdot$ odd

odd + odd = even

if $a + b =$ even, $c + d \cdot$ even

even + even = even

Question 8 continued

∴ will always produce even no.

∟ can eliminate 9, 27 as being possible values of $a+b+c+d$

36 can only be produced if $a,b,c,d = 9$

However, since b is smaller, $a+b+c+d < 36$

∴ can eliminate 36

$\therefore a+b+c+d = 18$ since $a+b = c+d$ replace

$a+b+a+b = 18$

$2a+2b = 18$ ÷2

$a+b = 9$ //

C. $6a+2b+3c+d \equiv 3 \pmod 7$

$4a+2a+2b+2c+c+d \equiv 3 \pmod 7$

$4a+2(a+b)+2c+(c+d) \equiv 3 \pmod 7$

$4a+2(9)+2c+(9) \equiv 3 \pmod 7$

$4a+18+2c+9 \equiv 3 \pmod 7$

$4a+2c+27 \equiv 3 \pmod 7$

$2c \equiv -4a-24 \pmod 7$

$-4a-24 \pmod 7 = 3a-3 \pmod 7$

$\therefore 2c \equiv 3a-3 \pmod 7$

$2c \equiv 3(a-1) \pmod 7$

×4

$8c \equiv 12(a-1) \pmod 7$

$12a-12 \pmod 7 = 5a-5 \pmod 7$

$= 5(a-1) \pmod 7$

$8c \equiv 5(a-1) \pmod 7$ if $a \equiv b \pmod m$ then $b \equiv a \pmod m$

$5(a-1) \equiv 8c \pmod 7$

$8c \pmod 7 = c \pmod 7$

$\therefore 5(a-1) \equiv c \pmod 7$

$c \equiv 5(a-1) \pmod 7$ //

Question 8 continued

a. If $b<a$, and $a+b=9$, $a \geq 5$

try: $a=5$ $b=4$ $c=6$ $d=3$ $d<b$ meant to be $b<d$ ✗

$a=6$ $b=3$ $c=4$ $d=5$ ✓

$a=7$ $b=2$ $c=2$ $b=C$ meant to be $b<c$ ✗

$a=8$ $b=1$ $c=0/7$ ← can't be 7 since must be even, can't be 0 since $c<b$ — not true ✗

$a=9$ $b=0$ $c=5$ ← should be even ✗

$\therefore n = 6345$